SHAPE THINKING AND STUDENTS' ACTIVITY WITH SIMULATIONS AND TABLES

Toni York
Montclair State University
yorka1@montclair.edu

Nicole Panorkou
Montclair State University
panorkoun@montclair.edu

The construct of static and emergent shape thinking (Moore & Thompson, 2015) characterizes differences in students' reasoning about graphs. In our previous work with middle school students, we found that this construct may also be useful in characterizing students' reasoning about other representations such as simulations and tables. In this paper, we present data from six students' reasoning to initiate a discussion around the possible nature of static and emergent shape thinking in the context of simulations and tables that would contribute to an expansion of the current framework to include these representations.

Keywords: Design Experiments, Mathematical Representations, Middle School Education, Technology

Covariational reasoning involves the coordination of simultaneous changes in two related quantities that are varying together (Thompson & Carlson, 2017). A quantity is defined as "someone's conceptualization of an object such that it has an attribute that could be measured" (Thompson & Carlson, 2017, p. 425). These quantities need not be numerical. For example, a student may reason covariationally only about the direction of change (Carlson et al., 2002), such as an observation that "as *X* increases, *Y* decreases." Many researchers have also characterized students' forms of covariational reasoning in different ways. For instance, reasoning in which a student shows a chunky rather than a smooth image of change (Castillo-Garsow et al., 2013). These forms of reasoning are crucial for students' development of several mathematical concepts, including the understanding of functions (Thompson & Carlson, 2017). In addition to mathematics, students' covariational reasoning has also been studied in science education since relationships between two or more covarying quantities are also prevalent in the study of many science topics. For example, we have found that middle school students can reason covariationally in the context of various scientific phenomena such as gravity (Basu et al., 2020; Panorkou & Germia, 2021), weather (York et al., 2021), and the greenhouse effect (Basu & Panorkou, 2019).

To support students' covariational reasoning in these scientific contexts, we iteratively developed and tested module designs that integrate mathematics and science content through interactive simulations as well as accompanying graphing tasks and questioning (Panorkou & York, 2020). As we examined the data from these design experiment iterations, one of the frameworks we used to characterize students' reasoning in the graphing tasks was Moore and Thompson's (2015) idea of static versus emergent shape thinking. A student engages in static shape thinking when they reason about a graph as an object in and of itself, such as a wire or a hill whose properties are not connected to an underlying covariational meaning. In contrast, emergent shape thinking involves reasoning about a graph as a meaningful record or emerging trace of the relationship between covarying quantities. Furthermore, emergent reasoning also includes recognizing the same underlying mathematical structures across different graphical representations (Moore, 2021) such as different coordinate systems (Paoletti et al., 2018).

While Moore and Thompson (2015) used the *shape thinking* construct to characterize students' thinking in graphs, as we analyzed our data, we noticed that this construct also seemed relevant to describe students' reasoning in other contexts. Specifically, in our previous work with our Climate Module (Germia et al., 2022; Panorkou et al., 2022), we used the distinction between static and emergent shape thinking to show progressions in two pairs of students' reasoning as they transitioned between working with different artifacts including a simulation, data tables, and graphs. In doing so, we found that these students reasoned both about the simulation and tables as well as the graphs in ways that could be characterized as static or emergent. We therefore conjectured that this framework might be expanded to include these other types of artifacts.

To explore this possible expansion of the framework of static and emergent shape thinking, we conducted a further analysis of our Climate Module data. Specifically, we explored the research question: *What forms of reasoning that can be characterized as shape thinking do students exhibit as they interact with different artifacts?*

Methods

To explore this research question, we analyzed data collected from the first iteration of a whole-class design experiment (DE) (Cobb et al., 2003) conducted in a sixth-grade classroom in the Northeast of the U.S. The DE focused on an instructional module we designed that aimed to support students in reasoning covariationally about the relationship between latitude and temperature in the Earth's different climatic zones.

First, students were asked to explore the Climatic Zones simulation (Figure 1). The simulation shows how latitude and temperature covary by displaying these quantities in visual and textual readouts. These readouts change as the student moves the arrow on the right side of the screen to different locations with their mouse. The locations are spread in the three main climatic zones: tropical, temperate, and polar. Latitude is displayed using both positive and negative numbers, where positive latitudes are located in the north and negative latitudes are located in the south. As the distance away from the Equator (the quantity measured by latitude) increases either to the north or to the south, the temperature tends to decrease.

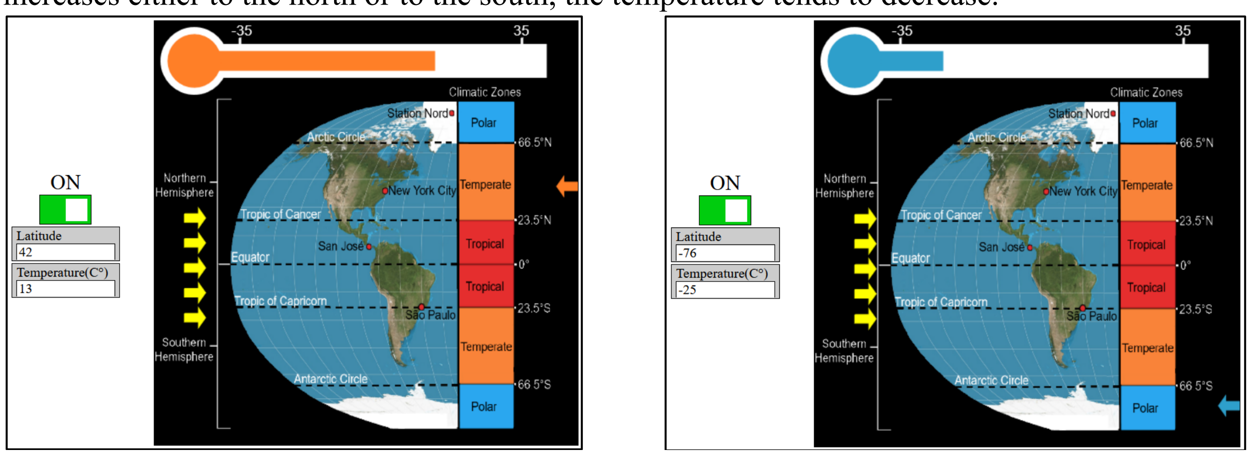

Figure 1: The Climatic Zones Simulation

The accompanying task involved creating tables and graphs of the latitude and temperature data found in this simulation (Figure 2). We also designed questioning to elicit students' reasoning about these quantities and the ways in which the different artifacts represent the

relationship between them. For example, we asked questions such as, "What did you notice in the simulation?", "What patterns do you see in your table?", and "What does your graph show?"

The class met for two 25-minute sessions via Google Classroom, during which the students were placed in pairs in breakout rooms by the teacher. Three of these pairs were selected by the teacher to be interviewed by the researchers. As each of the selected pairs worked on the module in their breakout room, a researcher joined them to observe their work and follow their reasoning. The six students' video, text chat, and shared screens during these breakout room sessions were recorded, and these recordings were transcribed for analysis. The transcriptions include speech, gestures, and screen/mouse actions.

Latitude	Air Temperature
-67	- 19
- 47	12
-23	23
-10	24
0	35
10	31
23	26
47	8
67	- 9

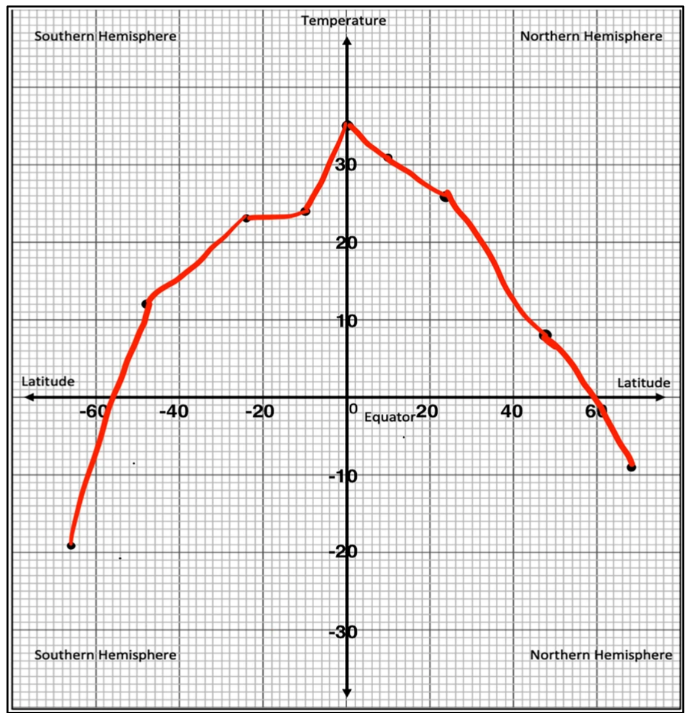

Figure 2: Recreation of a Student Table and Graph

In this paper we present the retrospective analysis of the activity of all six students: Ali, Jaden, Mikhail, Tasif, Jami, and Gaelyn. This analysis proceeded in two different stages. In the first stage, we identified and coded relevant excerpts of the transcripts in which the students reasoned about either the simulation, a table, or a graph. We use the term 'relevant' here to mean any reference to the features of an artifact, conversations about what an artifact shows, and any mention of the quantities or relationships represented in an artifact. As a counterexample, logistical conversations about opening the simulation or the task documents were not coded as relevant to an artifact. Then, in the second stage, we analyzed the identified excerpts again to code them as examples of the students displaying either static or emergent shape thinking.

Findings

In this section we organize our findings by the type of artifact the students were reasoning about and describe the excerpts that we characterized as showing static or emergent shape thinking in each case. Although the students did reason both statically and emergently with their

graphs (Germia et al., 2022; Panorkou et al., 2022), we decided to omit the familiar context of graphs here for brevity and focus instead on the simulation and table.

Static Shape Thinking with a Simulation

After exploring the Climatic Zones simulation freely for a few minutes, the students were asked what they had noticed. In response, Ali, Jami, and Tasif talked about the three zones. For instance, Ali responded, “The earth is divided into three different zones. The tropical zone, the temperate zone, and the polar zone.” Similarly, Jami stated, “The three major zones are polar, temperate, and tropic.” Tasif also commented that, “What I noticed in the simulation is that the arrow on the right, it points to a certain latitude. So you could change it and there are zones like polar, temperate, and tropical.” In these excerpts, all three students reasoned about visual features of the simulation, thinking of it as showing a map of the Earth and its different regions.

While Ali and Jami’s reasoning illustrated a static image of that map, Tasif was able to identify latitude as a quantity that varied. However, at this stage he did not reason about how both latitude and temperature covary. Like Tasif, Gaelyn also focused on the variation of latitude. She described the Equator as a separator between regions, saying, “From the Equator up is positive [latitude] and from the Equator down is negative [latitude].” She also explained that “the Equator is at the 0-degree [latitude] line between the northern and southern hemisphere.”

Mikhail described the simulation’s arrow and the different zones as well. However, he focused on the changing temperature in these different locations:

> Mikhail: This simulation, every time you move the arrow, it shows you the polar, the temperate, or the tropical zones and the things that separate them [moving mouse to point to these different locations on the simulation]. And it tells you every time you move to a certain area, it tells you the heat.

Similar to Tasif and Gaelyn, Mikhail focused on the variation of one quantity. Likewise, Jaden observed that the Arctic and Antarctic Circles are “what separates the polar regions from the temperate regions.” He also noted changes in temperature similarly to Mikhail, stating where it was negative or positive: “In the polar regions, the temperatures are negative. And in the temperate and tropical regions the temperatures are positive.” Mikhail and Jaden’s discussions of the changing temperature note only that this quantity is different in different places, not what the relationship is between the latitudes of those places and their temperatures.

All six students illustrated a simulation-as-map reasoning, which did not involve connections to an underlying covariational meaning of these features, even when the students made note of changes in latitude or temperature as they controlled the simulation’s arrow. We therefore argue that these examples can be characterized as showing the students’ static shape thinking as they reasoned about the simulation.

Emergent Shape Thinking with a Simulation

As they were asked questions about what was changing in the simulation and what patterns they saw in those changes, the students also spoke about the two varying quantities together. When Gaelyn was asked to order the cities in the simulation from warmest to coldest, she stated:

> Gaelyn: Because San Jose is right in the middle of the tropical, and then São Paulo is right on the line of the tropical and the temperate, and then New York City is in the temperate, and then Station Nord is in the polar. So that would be the order of it.

Gaelyn’s statement shows that she was able to describe how temperature changes as you move from zone to zone. Jami also illustrated similar reasoning:

Jami: The polar zone is actually very cold because it is up north. And the temperate zone, since it is right between the Equator where it is most hottest and the polar zone, it is pretty warm, like the average temperature. And the tropical zones are very hot because they are more close to the Equator.

Both Gaelyn and Jami's statements may be interpreted as illustrating covariational reasoning because they reasoned that both quantities change in relation to each other. While one might describe Gaelyn's and Jami's reasoning as illustrating change in chunks (zones), Jami's last statement about tropical zones being hot because they are "more close" to the Equator shows evidence of possibly having constructed a smooth image of change.

Mikhail showed a smooth image of change in his reasoning by describing the changes in temperature as a continuous movement from zone to zone:

Mikhail: The closer you go to the tropical, the hotter it is [moving mouse slowly from the northern polar zone to the tropical zone]. The closer you go to the polar, the colder it is [moving mouse from the tropical zone to the southern polar zone]. And the temperate is more in the middle. And, and could go each way [moving mouse within the southern temperate zone].

Mikhail identified that this relationship "could go each way" although he described this continuous change as occurring zone by zone. In contrast, Jaden and Ali showed that they constructed a smooth image of change in latitude and temperature. For example, Jaden noted that "The higher the latitude the colder it is going to be." Similarly, Ali stated, "The closer you are to the Equator the hotter it is."

Of the six students, Tasif illustrated the most sophisticated form of covariational reasoning. Looking at the two hemispheres separately, he stated the relationship he saw between movement towards the Equator and an increase in temperature, saying, "In the northern hemisphere, the more south you go, [it] gets hotter. But if in the southern hemisphere, the more north you go, it'll be a little hotter until we reach 0 latitude." Furthermore, Tasif also reasoned about a pattern he noticed in the rate of this change in different zones: "When you're at the temperate zone, it slowly decreases as you can see in simulation [moving mouse within temperate zones in both hemispheres], but once you reach the polar zones [moving mouse within polar zones], it starts decreasing rapidly."

In sum, all six students made a connection between changes in latitude (distance from the Equator) and changes in temperature in the simulation. In contrast to the static shape thinking examples in the previous section, each of these examples show a student reasoning about the simulation as simultaneously representing the changes in both quantities. While students' forms of covariational reasoning might be different, they involved simulation-as-relationship thinking rather than simulation-as-map thinking. Because of this difference, we thus interpret these excerpts as showing the students' emergent shape thinking as they reasoned about the simulation.

Static Shape Thinking with a Table

Only three of the six students provided data on their reasoning about their tables. Two students focused more on graphing during their interviews and no audio was recorded for one student during this part of the interview. Of the remaining three students, Ali and Mikhail both answered questions about patterns in their tables with descriptions of the numbers that they could see. Ali stated, "I think it's like the same exact numbers, just negative [moving mouse up and down the latitude column]." He then further described this visual pattern he had found in the

latitude column by adding, “So here’s 0 [placing mouse on the entry for 0 latitude]. Then you’ve got -10 and 10, 23 and -23, -47, 47, 67, and -67 [moving mouse up and down to each latitude value as he reads them out].” Similarly, Mikhail noted, “there is some type of pattern repeating. […] when the latitude is -23, the air temperature is 23. And when the latitude was 23, it [temperature] was 26. It is kind of similar.” He later also described how he saw the temperature changing in his table by discussing these changes as a rising and falling shape.

Mikhail: The temperature, I say it would rise up from negative [latitude], I mean, it would become very low for negative [latitude] since -19 [°C]. […] But then when it reaches -47 latitude, the temperatures start rising. And for the next, it’s like a slowly curved rising [tracing a hill-like motion with his hand]. And then it slopes down again [moves from the peak to the original starting height].

Ali’s description of the symmetry in the numbers of the latitude column and Mikhail’s description of the repeated appearance of 23 and a “kind of similar” 26 in the table are both examples of visual patterns in the appearance of the table’s numbers (Figure 3).

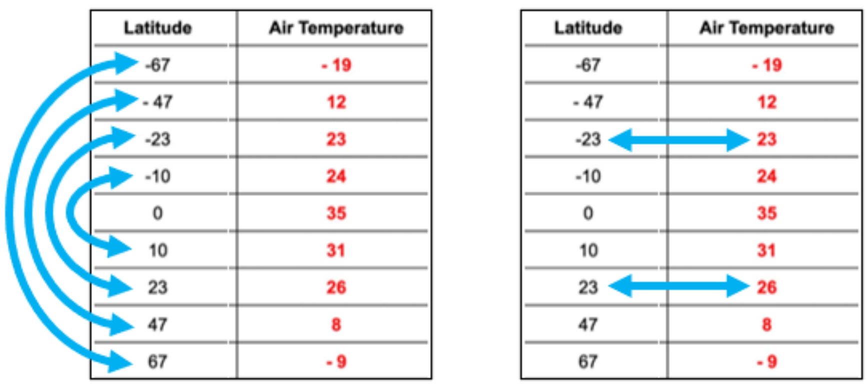

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Figure 3: Ali’s (left) and Mikhail’s (right) Visual Patterns

Neither of these patterns carry any meaning associated with the covariational relationship between latitude and temperature, but are rather a result of how the data was chosen in the design of the simulation and how the given latitude values were chosen for the table creation task. Similarly, although Mikhail mentioned the temperature changing for different values of latitude in the table, he reasoned about the visual up-and-down shape of these changing values without connecting this to the underlying covariational relationship that it represents. These excerpts thus show examples of the students’ static shape thinking as they reasoned about the table.

Emergent Shape Thinking with a Table

Similar to the section before, we focus here on the analysis of the data of only the three students. When asked what his table told him, Ali responded:

Ali: Yeah, so the Equator’s at 0 [pointing to the row with 0 latitude on table]. And then like, the temperature’s dropping faster. Like the negatives [moving mouse from 0 latitude to the top of the table] are the south of it and they’re dropping faster [moving mouse along the temperature values for the south in the top half of the table] than the north of the Equator [moving mouse along the bottom half of the table].

Ali’s reasoning shows a description of how the rate of change of the temperature was different in the southern and northern latitudes. His reasoning about the differing rate of change

of temperature across different regions of latitude implies that he saw the table as representing the simultaneous covariation of these quantities in order to make an observation about how this covariation itself varies. He also later directly stated the relationship he saw in his table between latitude and temperature, saying,

> Ali: "The farther from 0 [moving mouse from 0 latitude to the bottom of the table], like, the colder. And even that, I think this [moving mouse from 0 latitude to the top of the table] is just the same."

This later statement about how the temperature is colder the farther from 0 in both directions on the table makes his reasoning more explicit. For Ali, in these excerpts, the table records how the quantities are changing in relationship with each other.

Similarly, Tasif found a relationship between distance from the Equator and temperature in his table, noting, "the farther you go from the Equator, the colder it will be." Mikhail then replied to Tasif's observation by adding, "because there's less direct sunlight [farther from the Equator]." Tasif's statement of this relationship also shows that he imagined the table as a record of distance from the Equator (latitude) and temperature varying simultaneously. Mikhail's reply to Tasif's statement also implies a similar way of thinking about the table, offering a causal relationship between the three simultaneously varying quantities of latitude, amount of direct sunlight, and temperature.

Each of these excerpts involve a student reasoning about what the numbers in the table meaningfully represent in terms of the underlying covarying quantities, seeing the table as a record of a covariational relationship. We therefore interpret these excerpts to exemplify students' emergent shape thinking as they reasoned about the table.

Concluding Remarks

The analysis showed that as the students worked with the simulation and the table, their reasoning varied in ways that can be characterized in terms of shape thinking. We interpret their reasoning to illustrate evidence of both static and emergent thinking at different points during the DE. Specifically, all six students reasoned both statically and emergently with the simulation and two of them, Ali and Mikhail, also reasoned in both ways with the table. Table 1 presents the resulting expansion of the Moore and Thompson (2015) framework with definitions and examples of each type of shape thinking taken from the data presented in this paper.

Table 1: Expansion of the Moore and Thompson (2015) Shape Thinking Framework

	Static Shape Thinking	**Emergent Shape Thinking**
Simulation	*Reasoning about simulation features without connecting them to an underlying covariational meaning.* "The earth is divided into three different zones. The tropical zone, the temperate zone, and the polar zone."	*Reasoning about how simultaneously varying quantities in the simulation are related to each other.* "The closer you are to the Equator the hotter it is."
Table	*Reasoning about a visual*	*Reasoning about numerical*

	pattern of numbers without connecting this to an underlying covariational meaning. "I think it's like the same exact numbers, just negative [moving mouse up and down the latitude column]."	*patterns that describe a covariational relationship.* "The farther from 0 [moving mouse from 0 latitude to the bottom of the table], like, the colder. And even that, I think this [moving mouse from 0 latitude to the top of the table] is just the same."

Furthermore, we found that all six students recognized the same underlying covariational structure in their emergent reasoning with both the simulation and graph (graph data reported in Germia et al. (2022) and Panorkou et al. (2022)). Of the three students for whom we also have table data (Ali, Mikhail, and Tasif), all three of them also reasoned emergently with the table and therefore recognized the same structure across all three artifacts. This implies that Moore's (2021) discussion of transfer in graphical shape thinking may also be expanded to include different types of artifacts other than graphs.

This expansion of the shape thinking construct allows researchers to analyze students' reasoning with this lens across multiple representations, broadening its use and offering a way to discuss progressions in students' reasoning as they transition between artifacts. In practice, this expansion could also contribute to teachers' understanding of their students' reasoning, allowing them to more closely tailor their instructional moves to a given student's current thinking. For example, it is important to note that students may reason both statically and emergently at different times with the same artifact. Reasoning statically in one instance does not mean the student does not see the emergent relationship, while reasoning emergently in one instance also does not mean they are necessarily thinking about that relationship at a different time or in a different context.

Further study is needed into the type of questioning that leads to specific expression of shape thinking, acknowledging that students may be reasoning emergently but not showing evidence of this because the researcher's questioning does not elicit it. Future work may also consider data from other DEs to see if other students reasoned similarly and explore patterns in the ways that students use shape thinking when working with representations other than graphs. Finally, it would also be interesting to explore what other artifacts or representations of covariation the shape thinking framework might be expanded to include.

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